

# Higher dimensional flat embeddings of black strings in (2+1) dimensions

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## Abstract

We obtain (3+1) and (3+2) dimensional global flat embeddings of (2+1) uncharged and charged black strings, respectively. In particular, the charged black string, which is the dual solution of the Banados-Teitelboim-Zanelli black holes, is shown to be embedded in the same global embedding Minkowski space structure as that of the (2+1) charged de Sitter black hole solution.

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It has been well-known that a thermal Hawking effect on a curved manifold [1] can be looked at as an Unruh effect [2] in a higher flat dimensional space time. Recently, the isometric embeddings of the Reissner-Nordström (RN) black hole [3] and M2-, D3-, M5-branes [4] into flat manifolds with two times have been studied to yield some insight into the global aspects of the space-time geometries in the context of brane physics. Following the global embedding Minkowski space (GEMS) approach [5,6], several authors [3,7–10] recently have shown that this approach could yield a unified derivation of temperature for various curved manifolds such as rotating Banados-Teitelboim-Zanelli (BTZ) [11–14], Schwarzschild [15] together with its anti-de Sitter (AdS) extension, RN [16] and RN-AdS [9]. Historically, the higher dimensional global flat embeddings of the black hole solutions are subjects of great interest to mathematicians as well as physicists. In differential geometry, it has been well-known that the four dimensional Schwarzschild metric is not embedded in  $R^5$  [17]. Very recently, Deser and Levin firstly obtained (5+1) dimensional global flat embeddings of (3+1) Schwarzschild black hole solution [3,7]. Moreover, the (3+1) dimensional RN-AdS, RN and Schwarzschild-AdS black holes are shown to be embedded in (5+2) dimensional GEMS manifolds [9].

On the other hand, the static, rotating and charged (2+1) dimensional BTZ AdS black holes are shown to have (2+2), (2+2) and (3+3) GEMS structures [3,7,10], while the static, rotating and charged (2+1) dimensional de Sitter (dS) black holes are shown to have (3+1), (3+1) and (3+2) GEMS structures, respectively [10]. Recently, the (2+1) dimensional BTZ black hole [11] has attracted much attention as a useful model for realistic black hole physics [12] in spite of the fact that the space-time curvature is constant. Moreover, it has been discovered the novel aspects that the thermodynamics of higher dimensional black holes can often be interpreted in terms of the BTZ solution [18], and a slightly modified solution of the BTZ black hole yields an solution to the string theory, so-called the black string [19]. Here one notes that this black string solution is in fact only a solution to the lowest order  $\beta$ -function equation to receive quantum corrections [20]. It is now interesting to study the geometry of (2+1) dimensional black strings, which are the dual solutions of the BTZ black

holes, and their thermodynamics through further investigation, together with those of the BTZ solutions.

In this Brief Report we will analyze Hawking and Unruh effects of the (2+1) dimensional black strings in terms of the GEMS approach. First, we briefly recapitulate the known rotating (2+1) BTZ black hole and the corresponding charged black string. Next, we will treat the global higher dimensional flat embeddings of the (2+1) uncharged and charged black strings. In particular, we will show that the charged black string is embedded in (3+2) GEMS structure.

Now we briefly summarize the duality properties of the black strings given in Ref. [19]. These black string solutions are dual ones for the well known (2+1) dimensional rotating BTZ black holes which are described by the 3-metric, antisymmetric tensor and dilaton as follows

$$\begin{aligned} ds^2 &= -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2, \\ B_{\phi t} &= \frac{r^2}{l}, \\ \Phi &= 0. \end{aligned} \tag{1}$$

Here the lapse and shift functions are given as

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi = -\frac{J}{2r^2}, \tag{2}$$

respectively. Note that for the nonextremal case there exist two horizons  $r_\pm(J)$  satisfying the following equations,

$$0 = -M + \frac{r_\pm^2}{l^2} + \frac{J^2}{4r_\pm^2}, \tag{3}$$

respectively. Without solving these equations explicitly, one can then rewrite the mass  $M$  and angular momentum  $J$  in terms of these outer and inner horizons as follows

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+ r_-}{l}. \tag{4}$$

Furthermore these relations can be used to yield the lapse and shift functions of the forms

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2}, \quad N^\phi = -\frac{r_+ r_-}{r^2 l}, \quad (5)$$

respectively. Here one notes that this BTZ space originates from AdS one via the geodesic identification  $\phi = \phi + 2\pi$ .

On the other hand, one can obtain dual solution [21] through the transformation

$$\begin{aligned} g_{\phi\phi}^d &= g_{\phi\phi}^{-1}, \quad g_{\phi\alpha}^d = B_{\phi\alpha} g_{\phi\phi}^{-1}, \quad g_{\alpha\beta}^d = g_{\alpha\beta} - (g_{\phi\alpha} g_{\phi\beta} - B_{\phi\alpha} B_{\phi\beta}) g_{\phi\phi}^{-1}, \\ B_{\phi\alpha}^d &= g_{\phi\alpha} g_{\phi\phi}^{-1}, \quad B_{\alpha\beta}^d = B_{\alpha\beta} - 2g_{\phi[\alpha} B_{\beta]\phi} g_{\phi\phi}^{-1}, \\ \Phi^d &= \Phi - \frac{1}{2} \ln g_{\phi\phi}, \end{aligned} \quad (6)$$

where  $\alpha, \beta$  run over all directions except  $\phi$ , and  $B_{\phi t} = r^2/l$ . The corresponding dual solution is then given as

$$ds_d^2 = -N^2 dt^2 + N^{-2} dr^2 + \frac{1}{r^2} (d\phi_d + N_d^\phi dt)^2, \quad (7)$$

where the shift function, antisymmetric tensor and dilaton are defined as

$$\begin{aligned} N_d^\phi &= B_{\phi t} = \frac{r^2}{l}, \\ B_{\phi t}^d &= N^\phi = -\frac{J}{2r^2}, \\ \Phi^d &= -\ln r, \end{aligned} \quad (8)$$

respectively. Here one notes that  $\phi_d/r^2$  is periodic such that  $\phi_d/r^2 = \phi_d/r^2 + 2\pi$  and, as expected, the Hawking temperature [1] and entropy are dual invariant

$$T_H^d = T_H = \frac{r(r_+^2 - r_-^2)}{2\pi r_+ l (r^2 - r_+^2)^{1/2} (r^2 - r_-^2)^{1/2}}, \quad (9)$$

$$S_{BH}^d = S_{BH} = 2\pi r_+. \quad (10)$$

Now one can diagonalize the above dual metric with the coordinate transformation

$$t = \frac{l(\hat{x} - \hat{t})}{(r_+^2 - r_-^2)^{1/2}}, \quad \phi_d = \frac{r_+^2 \hat{t} - r_-^2 \hat{x}}{(r_+^2 - r_-^2)^{1/2}}, \quad r^2 = l\hat{r}. \quad (11)$$

The (2+1) dimensional charged black string solution [22] is then given by the 3-metric, antisymmetric tensor and non-vanishing dilaton as follows

$$\begin{aligned}
d\hat{s}^2 &= -\left(1 - \frac{\mathcal{M}}{\hat{r}}\right) d\hat{t}^2 + \left(1 - \frac{\mathcal{Q}^2/\mathcal{M}}{\hat{r}}\right) d\hat{x}^2 + \left(1 - \frac{\mathcal{M}}{\hat{r}}\right)^{-1} \left(1 - \frac{\mathcal{Q}^2/\mathcal{M}}{\hat{r}}\right)^{-1} \frac{l^2 d\hat{r}^2}{4\hat{r}^2}, \\
\hat{B}_{\hat{x}\hat{t}} &= \frac{\mathcal{Q}}{\hat{r}}, \\
\hat{\Phi} &= -\frac{1}{2} \ln \hat{r} l,
\end{aligned} \tag{12}$$

where  $\mathcal{M} = r_+^2/l$  and  $\mathcal{Q} = J/2$ . Here note that the charge of the black string is linearly proportional to the angular momentum of the rotating BTZ black hole.

Next, in order to construct the GEMS structures of the black strings, we consider the uncharged black string 3-metric

$$d\hat{s}^2 = -\left(1 - \frac{\mathcal{M}}{\hat{r}}\right) d\hat{t}^2 + d\hat{x}^2 + \left(1 - \frac{\mathcal{M}}{\hat{r}}\right)^{-1} \frac{l^2 d\hat{r}^2}{4\hat{r}^2}. \tag{13}$$

After algebraic manipulation, we obtain the (3+1) dimensional GEMS  $d\hat{s}^2 = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2$  given by the coordinate transformations with only one additional space-like dimension  $z^3$ ,

$$\begin{aligned}
z^0 &= l \left(1 - \frac{\hat{r}_H}{\hat{r}}\right)^{1/2} \sinh \frac{\hat{t}}{l}, \\
z^1 &= l \left(1 - \frac{\hat{r}_H}{\hat{r}}\right)^{1/2} \cosh \frac{\hat{t}}{l}, \\
z^2 &= \hat{x}, \\
z^3 &= -l \left(1 + \frac{\hat{r}_H}{\hat{r}}\right)^{1/2} + l \ln \left(\hat{r}^{1/2} + (\hat{r} + \hat{r}_H)^{1/2}\right),
\end{aligned} \tag{14}$$

where  $\hat{r}_H = \mathcal{M}$ . We will later show how to construct the GEMS structure of the more general case of charged black string, which can yield the above result (14) in the uncharged limit. Here note that the embedding dimension of this solution is the same as that of the (2+1) uncharged dS case [23], dual version of BTZ black hole. For the trajectories, which follow the Killing vector  $\xi = \partial_{\hat{t}}$  on the uncharged black string manifold described by  $(\hat{t}, \hat{r}, \hat{x})$ , one can obtain constant 3-acceleration

$$\hat{a} = \frac{\hat{r}_H}{l\hat{r} \left(1 - \frac{\hat{r}_H}{\hat{r}}\right)^{1/2}}. \tag{15}$$

We see how the uncharged black string solution yields a finite Unruh area due to the periodic identification of  $\phi_d/(l\hat{r}) \bmod 2\pi$ . The Rindler horizon condition  $(z^1)^2 - (z^0)^2 = 0$  implies  $\hat{r} = \hat{r}_H$  and the remaining embedding constraints yield  $z^2 = \hat{x}$  and  $z^3 = f(r)$  where  $f(r)$  can be read off from Eq. (14). The area of the Rindler horizon [24] yields the entropy of the uncharged black string

$$\begin{aligned}\hat{S} &= \int_{-\pi(l\hat{r}_H)^{1/2}}^{\pi(l\hat{r}_H)^{1/2}} dz^2 dz^3 \delta(z^3 - f(r)) \\ &= \int_{-\pi(l\hat{r}_H)^{1/2}}^{\pi(l\hat{r}_H)^{1/2}} dz^2 = 2\pi\hat{r}_H.\end{aligned}\tag{16}$$

Now we construct the GEMS structure of the charged black string. In order to obtain the  $d\hat{t}^2$  term in Eq. (12) we make an ansatz of two coordinates  $(z^0, z^1)$  in Eq. (20) to yield

$$-(dz^0)^2 + (dz^1)^2 = -\left(1 - \frac{\hat{r}_H}{\hat{r}}\right) d\hat{t}^2 + \frac{1}{4} \left(\frac{l^2 \hat{r}_H}{\hat{r}_H - \hat{r}_-}\right) \left(\frac{\hat{r}_H}{\hat{r}^2}\right)^2 \left(1 - \frac{\hat{r}_H}{\hat{r}}\right)^{-1} d\hat{r}^2 \tag{17}$$

where  $\hat{r}_H = \mathcal{M}$ , and  $\hat{r}_- = \mathcal{Q}^2/\mathcal{M}$ . Similarly, the  $d\hat{x}^2$  term in Eq. (12) can be constructed by exploiting an ansatz of two coordinates  $(z^2, z^3)$  in Eq. (20) as follows

$$-(dz^2)^2 + (dz^3)^2 = \left(1 - \frac{\hat{r}_-}{\hat{r}}\right) d\hat{x}^2 - \frac{1}{4} \left(\frac{l^2 \hat{r}_-}{\hat{r}_H - \hat{r}_-}\right) \left(\frac{\hat{r}_-}{\hat{r}^2}\right)^2 \left(1 - \frac{\hat{r}_-}{\hat{r}}\right)^{-1} d\hat{r}^2. \tag{18}$$

Since the combination of Eqs. (17) and (18) yields

$$\begin{aligned}-(dz^0)^2 + (dz^1)^2 - (dz^2)^2 + (dz^3)^2 &= -\left(1 - \frac{\hat{r}_H}{\hat{r}}\right) d\hat{t}^2 + \left(1 - \frac{\hat{r}_-}{\hat{r}}\right) d\hat{x}^2 \\ &+ \left(1 - \frac{\hat{r}_H}{\hat{r}}\right)^{-1} \left(1 - \frac{\hat{r}_-}{\hat{r}}\right)^{-1} \frac{l^2 d\hat{r}^2}{4\hat{r}^4} \left[\hat{r}_H^2 + \hat{r}_H \hat{r}_- + \hat{r}_-^2 - \frac{\hat{r}_H \hat{r}_- (\hat{r}_H + \hat{r}_-)}{\hat{r}}\right] \\ &= d\hat{s}^2 - \frac{l^2 d\hat{r}^2}{4\hat{r}^3} (\hat{r} + \hat{r}_H + \hat{r}_-) \equiv d\hat{s}^2 - (dz^4)^2,\end{aligned}\tag{19}$$

we obtain the desired (3+2) GEMS  $d\hat{s}^2 = -(dz^0)^2 + (dz^1)^2 - (dz^2)^2 + (dz^3)^2 + (dz^4)^2$  for the charged black string [22] given by the coordinate transformations,

$$\begin{aligned}z^0 &= \left(\frac{l^2 \hat{r}_H}{\hat{r}_H - \hat{r}_-}\right)^{1/2} \left(1 - \frac{\hat{r}_H}{\hat{r}}\right)^{1/2} \sinh\left(\frac{\hat{r}_H - \hat{r}_-}{l^2 \hat{r}_H}\right) \hat{t}, \\ z^1 &= \left(\frac{l^2 \hat{r}_H}{\hat{r}_H - \hat{r}_-}\right)^{1/2} \left(1 - \frac{\hat{r}_H}{\hat{r}}\right)^{1/2} \cosh\left(\frac{\hat{r}_H - \hat{r}_-}{l^2 \hat{r}_H}\right) \hat{t},\end{aligned}$$

$$\begin{aligned}
z^2 &= \left( \frac{l^2 \hat{r}_-}{\hat{r}_H - \hat{r}_-} \right)^{1/2} \left( 1 - \frac{\hat{r}_-}{\hat{r}} \right)^{1/2} \cosh \left( \frac{\hat{r}_H - \hat{r}_-}{l^2 \hat{r}_-} \right) \hat{x}, \\
z^3 &= \left( \frac{l^2 \hat{r}_-}{\hat{r}_H - \hat{r}_-} \right)^{1/2} \left( 1 - \frac{\hat{r}_-}{\hat{r}} \right)^{1/2} \sinh \left( \frac{\hat{r}_H - \hat{r}_-}{l^2 \hat{r}_-} \right) \hat{x}, \\
z^4 &= -l \left( 1 + \frac{\hat{r}_H + \hat{r}_-}{\hat{r}} \right)^{1/2} + l \ln \left( \hat{r}^{1/2} + (\hat{r} + \hat{r}_H + \hat{r}_-)^{1/2} \right). \tag{20}
\end{aligned}$$

This equivalence between the (3+2) GEMS metric associated with Eq. (20) and the original curved space metric in Eq. (12) is the very definition of isometric embedding, mathematically developed by several authors [4,25]. Here one can also easily check that, in the uncharged limit  $\mathcal{Q} \rightarrow 0$  (or  $\hat{r}_- \rightarrow 0$ ), the above coordinate transformations  $(z^0, z^1, z^4)$  for the case of  $\mathcal{Q} \neq 0$  are exactly reduced to  $(z^0, z^1, z^3)$  for the case of  $\mathcal{Q} = 0$  in the previous ones (14), while  $(z^2, z^3)$  with  $\mathcal{Q} \neq 0$  merge into one positive definite coordinate  $z^2 (= \hat{x})$  with  $\mathcal{Q} = 0$  since in this uncharged limit  $-(dz^2)^2 + (dz^3)^2 \rightarrow d\hat{x}^2$  in Eq. (18). Note that this isometric embedding (20) is equivalent to the charged dS case [23], which is the dual version of charged BTZ black hole. For the trajectories, which follow the Killing vector  $\xi = \partial_{\hat{t}}$  on the charged black string manifold described by  $(\hat{t}, \hat{r}, \hat{x})$ , one can obtain constant 3-acceleration

$$\hat{a} = \frac{\hat{r}_H \left( 1 - \frac{\hat{r}_-}{\hat{r}} \right)^{1/2}}{l \hat{r} \left( 1 - \frac{\hat{r}_H}{\hat{r}} \right)^{1/2}}. \tag{21}$$

Now we see how the charged black string solution yields a finite Unruh area due to the periodic identification of  $\phi_d/(l\hat{r}) \bmod 2\pi$ . The Rindler horizon condition  $(z^1)^2 - (z^0)^2 = 0$  implies  $\hat{r} = \hat{r}_H$  and the remaining embedding constraints yield

$$\begin{aligned}
z^4 &= f(r), \\
(z^2)^2 - (z^3)^2 &= (l\mathcal{Q}/\mathcal{M})^2, \tag{22}
\end{aligned}$$

where  $f(r)$  can be read off from Eq. (20). Here one notes that such local and isometric embedding spaces have the global topology that cover only the area outside the horizon  $\hat{r}_H$  with the above embedding constraints (22) and cannot be extended to the entire solution [6,7].

With this global topology of the isometric embedding in mind, we can obtain the area of the Rindler horizon described as

$$\int dz^2 dz^3 dz^4 \delta(z^4 - f(r)) \delta\left([(z^2)^2 - (z^3)^2]^{1/2} - \frac{lQ}{\mathcal{M}}\right)$$

which, after performing trivial integrations over  $z^4$ , yields the desired entropy of the charged black string

$$\begin{aligned} \hat{S} &= \int_{-(lQ/\mathcal{M}) \sinh(\pi \hat{r}_H / (lQ^2/\mathcal{M})^{1/2})}^{(lQ/\mathcal{M}) \sinh(\pi \hat{r}_H / (lQ^2/\mathcal{M})^{1/2})} dz^3 \int_0^{[(z^3)^2 + (lQ/\mathcal{M})^2]^{1/2}} dz^2 \delta\left([(z^2)^2 - (z^3)^2]^{1/2} - \frac{lQ}{\mathcal{M}}\right) \\ &= \int_{-(lQ/\mathcal{M}) \sinh(\pi \hat{r}_H / (lQ^2/\mathcal{M})^{1/2})}^{(lQ/\mathcal{M}) \sinh(\pi \hat{r}_H / (lQ^2/\mathcal{M})^{1/2})} dz^3 \frac{lQ/\mathcal{M}}{[(lQ/\mathcal{M})^2 + (z^3)^2]^{1/2}} = 2\pi \hat{r}_H. \end{aligned} \quad (23)$$

In conclusion, we have investigated the higher dimensional global flat embeddings of (2+1) uncharged and charged black strings. These black strings are shown to be embedded in the (3+1) and (3+2)-dimensions for the uncharged and charged black strings, respectively. Here it is worthwhile noting that both the uncharged and charged black strings have the same GEMS structures as those of uncharged and charged de Sitter black holes, which are dual solutions of the corresponding BTZ anti-de Sitter black holes. Moreover, through further investigation, it will be interesting to study the topology of boundary surface of black holes associated with the nontrivial higher genus and the corresponding Yamabe invariant [26,27], in the framework of the GEMS structure.

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## REFERENCES

- [1] S.W. Hawking, Commun. Math. Phys. **42**, 199 (1975); J.D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); R.M. Wald, *Quantum Field Theory in Curved Space-time and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1994); J.D. Brown, C. Creighton, and R.B. Mann, Phys. Rev. D **50**, 6394 (1994).
- [2] W.G. Unruh, Phys. Rev. D **14**, 870 (1976); P.C.W. Davies, J. Phys. A **8**, 609 (1975).
- [3] S. Deser and O. Levin, Phys. Rev. D **59**, 0640004 (1999).
- [4] L. Andrianopoli, M. Derix, G.W. Gibbons, C. Herdeiro, A. Santambrogio and A. Van Proeyen, Class. Quantum Grav. **17**, 1875 (2000).
- [5] E. Kasner, Am. J. Math. **43**, 130 (1921); C. Fronsdal, Phys. Rev. **116**, 778 (1959); H.F. Goenner, *General Relativity and Gravitation* (Plenum, New York, 1980) Ed. A. Held; H. Narnhofer, I. Peter, and W. Thirring, Int. J. Mod. Phys. B **10**, 1507 (1996).
- [6] J. Rosen, Rev. Mod. Phys. **37**, 204 (1965).
- [7] S. Deser and O. Levin, Class. Quantum Grav. **14**, L163 (1997); Class. Quantum Grav. **15**, L85 (1998).
- [8] M. Beciu and H. Culetu, Mod. Phys. Lett. A **14**, 1 (1999); P.F. Gonzalez-Diaz, Phys. Rev. D **61**, 024019 (1999).
- [9] Y.-W. Kim, Y.-J. Park, and K.S. Soh, Phys. Rev. D **62**, 104020 (2000).
- [10] S.-T. Hong, Y.-W. Kim, and Y.-J. Park, Phys. Rev. D **62**, 024024 (2000); S.-T. Hong, W.T. Kim, Y.-W. Kim, and Y.-J. Park, Phys. Rev. D **62**, 064021 (2000).
- [11] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
- [12] S. Carlip, Class. Quantum Grav. **12**, 2853 (1995).

- [13] S.W. Kim, W.T. Kim, Y.-J. Park, and H. Shin, Phys. Lett. B **392**, 311 (1997).
- [14] D. Cangemi, M. Leblanc, and R.B. Mann, Phys. Rev. D **48**, 3606 (1993).
- [15] K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, KI. Math.-Phys. Tech., pp. 189-196 (1916).
- [16] H. Reissner, Ann. Physik **50**, 106 (1916); G. Nordstrom, Proc. Kon. Ned. Akda. Wet **20**, 1238 (1918); C.S. Peca and J.P.S. Lemos, Phys. Rev. D **59**, 124007 (1999); P. Mitra, Phys. Lett. B **459**, 119 (1999); S.W. Hawking and H.S. Reall, Phys. Rev. D **61**, 024014 (1999); B. Wang, E. Abdalla and R.-K. Su, Phys. Rev. D **62**, 047501 (2000).
- [17] M. Spivak, *A Comprehensive Introduction to Differential Geometry* (Publish or Perish, Berkeley, 1975) Vol 5, Chapter 11.
- [18] J. Maldacena and A. Strominger, JHEP **12**, 005 (1998); S. Hyun, J. Korean Phys. Soc. **33**, S532 (1998), hep-th/9704005; H.W. Lee and Y.S. Myung, Phys. Rev. D **58**, 104013 (1998); K. Sfetsos and K. Skenderis, Nucl. Phys. B **517**, 179 (1998).
- [19] G.T. Horowitz and D.L. Welch, Phys. Rev. Lett. **71**, 328 (1993); J.H. Horne, G.T. Horowitz, and A.R. Steif, Phys. Rev. Lett. **68**, 568 (1992).
- [20] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. B **262**, 593 (1985).
- [21] T. Buscher, Phys. Lett. B **201**, 466 (1988).
- [22] J. Horne and G.T. Horowitz, Nucl. Phys. B **368**, 444 (1992).
- [23] S. Deser and R. Jackiw, Ann. Phys. **153**, 405 (1984); A. Strominger and J. Maldacena, JHEP **02**, 014 (1998); M.-I. Park, Phys. Lett. B **440**, 275 (1998).
- [24] G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15**, 2738 (1977); R. Laflamme, Phys. Lett. B **196**, 449 (1987).
- [25] J. Nash, Ann. Math. **60**, 383 (1954); **63**, 20 (1956); C.J.S. Clake, Proc. R. Soc. London

A **314**, 417 (1970).

- [26] M. Anderson, *Comparison Geometry* (Cambridge University Press, 1997) Vol 30, pp49-82.
- [27] G.W. Gibbons, Class. Quantum Grav. **16**, 1677 (1999); G.T. Horowitz, *1992 Trieste Spring School on String Theory and Quantum Gravity*, hep-th/9210119; G.J. Galloway, K. Schleich, D. Witt and E. Woolgar, Phys. Rev. D **60**, 104039 (1999); M. Cai and G.J. Galloway, hep-th/0102149.